Quantum Raychaudhuri Equation

Saurya Das
University of Lethbridge
CANADA

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Plan

- Singularities and Raychaudhuri equation
- Quantum Raychaudhuri equation
- Implications for singularities
- Application: Cosmology

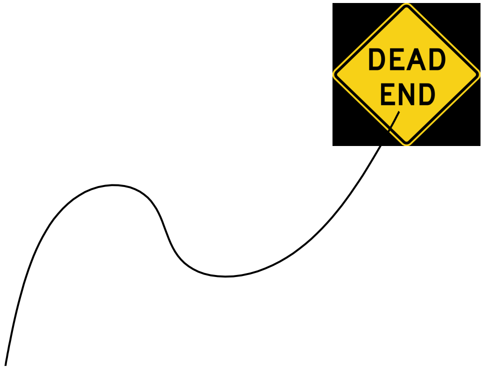
S. Das, Phys. Rev. **D89** 084068(2014) [arXiv:1311.6539]
A. F. Ali, S. Das, arXiv:1404.3093
S. Das, arXiv:1405.4011

Singularities in spacetime

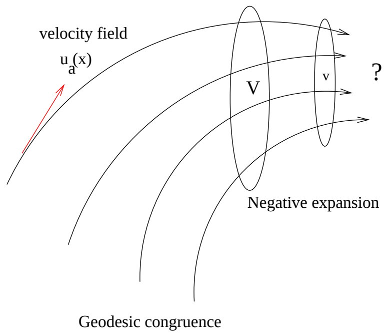
- Signal breakdown of General Relativity, start/end of time (big-bang/big-crunch), black holes, ...

- Curvature $\to \infty$ is indicative, but neither necessary, nor sufficient condition for singularities/pathological spacetime (coordinate dependent, many counterexamples)

- **Geodesic incompleteness** normally assumed to be necessary and sufficient condition for singular spacetimes



Raychaudhuri equation



Expansion $\theta = \frac{d \ln V}{d\lambda}$ (< 0 in above figure) (λ = affine parameter)

$$\frac{du_{a;b}}{d\lambda} = u_{a;b;c}\; u^c = \left[u_{a;c;b} + R_{cba}{}^d u_a\right] u^c \tag{1}$$

$$= \underbrace{\left(u_{a;c}u^c\right)_{;b}}_{= 0 \text{ (geodesic equation)}} - u^c{}_{;b}u_{a;c} + R_{cba}{}^d u^c u_d \tag{2}$$

$$= -u^c{}_{;b}u_{a;c} + R_{cbad}\; u^c u^d \; . \tag{3}$$

$h_{ab} = g_{ab} - u_a u_b$ (induced 3-metric)

$u_{a;b} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab} =$ Trace + Traceless symmetric + anti-symmetric

expansion shear twist

$$h^{ab}\,\frac{du_{a;b}}{d\lambda} = Tr\left(\frac{du_{a;b}}{d\lambda}\right) = Tr\left(-u^c{}_{;b}u_{a;c} + R_{cbad}\;u^c u^d\right)$$

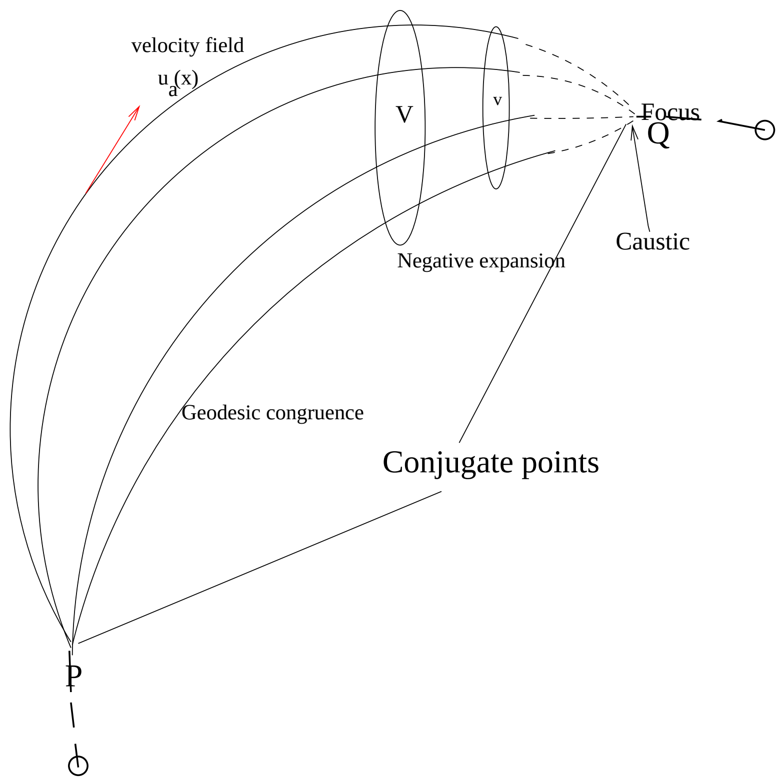
$$\frac{d\theta}{d\lambda} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \underbrace{\omega_{ab}\omega^{ab}}_{=0} - \underbrace{R_{cd}u^c u^d}_{>0} < 0$$

hypersurface orthog. *strong energy cond.*

If $\theta_0 = \theta(0) < 0$ (initially converging)

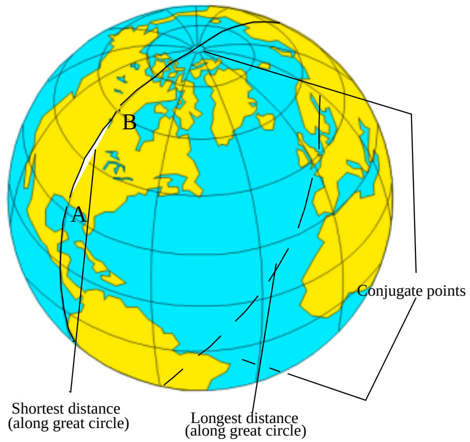
Focus/caustic for $\lambda \leq \frac{3}{|\theta_0|}$ *(finite proper time!)*

Raychaudhuri equation (1955) (also, Landau & Lifshitz)


velocity field
$u_a(x)$
V
v
Focus
Q
Caustic
Negative expansion
Geodesic congruence
Conjugate points
P

Singularity theorems

- Conjugate points due to Raychaudhuri equation.
- Geodesics are no longer maximal length curves.



- Maximal geodesics predicted by global arguments, on the other hand.
- Sufficiently long geodesics cannot exist. Geodesics are incomplete.
- Singularities! (Singularity theorems) - for most spacetimes
- R. Penrose, Phys. Rev. Lett. **14** (1965) 57-59; S. W. Hawking, R. Penrose, Proc. Roy. Soc. Lond. **A314** (1970) 529-548.

Note

- *Generality:* for all reasonable spacetimes (gravity universal & attractive)
- Fluid picture: velocity field $u^a(x)$
- But, classical $\to$ quantum? (Expectation values, Ehrenfest type theorem?)
- First, find a 'quantum velocity field'

Non-relativistic limit

$$u^a(x) \to v^a(\vec{x}, t) \quad {\scriptstyle (a\, =\, 1,2,3)}\ , \quad u^0 = 1\ , \ \lambda \to t$$

$$R_{cd} u^c u^d \to \nabla^2 V = 4\pi G \rho \geq 0,$$

$$\frac{d\vec{v}}{dt} = -\vec{\nabla} V(\vec{x}, t)$$

$$\frac{d\theta}{dt} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} - \nabla^2 V < 0 \ \leftarrow \text{(focusing in finite time)}$$

Quantum fluid picture

$$\psi(\vec{x},t) = \mathcal{R}e^{iS}$$

(Normalizable, single-valued, $\mathcal{R}$, S= Real.
E.g. complete set of H-atom bound states and scattering wavefunctions, $e^2/4\pi\epsilon_0 \to GMm$)

$$\vec{v}\,(\vec{x},t) = \frac{d\vec{x}}{dt} \equiv \frac{\hbar}{m}\,\mathcal{I}m\left(\frac{\vec{\nabla}\psi}{\psi}\right) = \frac{\hbar}{m}\,\vec{\nabla}S(\vec{x},t) \leftarrow$$ quantum velocity field!

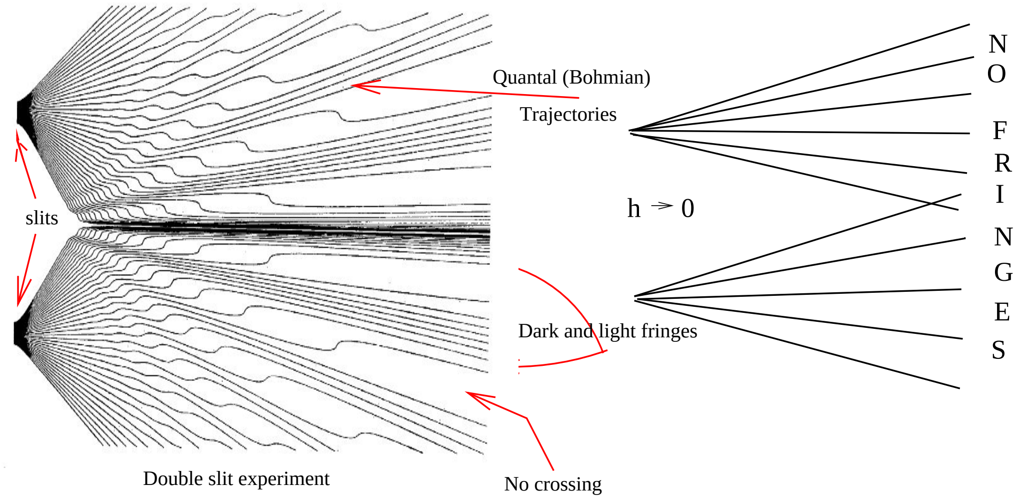
$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

Real and imaginary parts $\to$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$ (Probability conservation)

$$m\frac{d\vec{v}}{dt} = -m\vec{\nabla}V + \underbrace{\frac{\hbar^2}{2m}\vec{\nabla}\left(\frac{1}{\mathcal{R}}\nabla^2\mathcal{R}\right)}_{-\vec{\nabla}V_Q}$$ (Newton's law + quantum potential V_Q!)

- Initially, particles distributed as $\rho(0) = |\psi(0)|^2$ ('quantum equilibrium')
- Prob. conservation $\Rightarrow$ they remain distributed as $\rho(t) = |\psi(t)|^2$
- Each particle follows individual trajectories, subjected to $V + V_Q$
  quantal/Bohmian trajectories
- Make measurement: no need for collapse of wave-function



(C. Phillipides, C. Dewdney, B. J. Hiley, *Quantum Interference and the Quantum Potential*, il nuovo cimento B, **52**, no.1 (1979) 15-28)

- Stern-Gerlach, hydrogen atom, Aharonov-Bohm, harmonic oscillator, and all others $\to$ correct predictions
- Only dynamical input $\to$ Schrödinger equation
- Classical limit: trajectories $\to$ with $\hbar \to 0$ (interference disappears)
- L. de Broglie (1927), D. Bohm (1952), superconductivity, superfluidity, Bose-Einstein condensates (Gross-Pitaevskii equation)

Valid picture!

Quantum Raychaudhuri equation

$V \to V + V_Q/m$ in Raychaudhuri equation

$$\frac{d\theta}{dt} = -\frac{1}{3}\,\theta^2 - \sigma_{ab}\sigma^{ab} - \nabla^2 V + \underbrace{\frac{\hbar^2}{2m^2}\nabla^2\left(\frac{1}{\mathcal{R}}\nabla^2\mathcal{R}\right)}_{\text{quantum correction}}$$

quantum correction

attractive or repulsive?
focusing or defocusing?

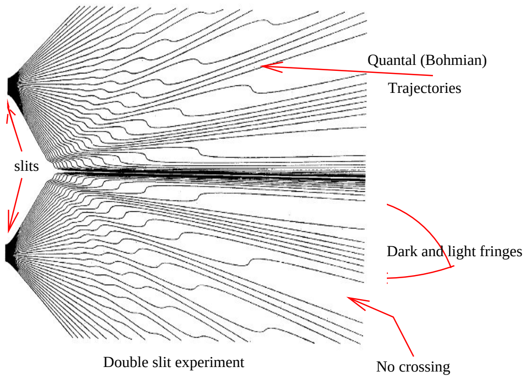
$$\psi = \psi_0 e^{-r^2/L^2} \;\to\; \frac{\hbar^2}{2m^2}\nabla^2\left(\frac{1}{\mathcal{R}}\nabla^2\mathcal{R}\right) \sim +\frac{1}{L^4} \;\leftarrow\; \textit{Repulsive at short distances}$$

$$-\frac{\hbar^2}{2m}\nabla^2\psi + [V + \underbrace{g|\psi|^2}_{\text{interaction}}]\psi = i\hbar\frac{\partial\psi}{\partial t} \quad \text{(Gross-Pitaevskii equation)}$$

$$\psi = \psi_0 \tanh(r/L\sqrt{2})\ (g > 0)\ ,\quad \psi = \sqrt{2}\ \psi_0\ \mathrm{sech}(r/L)\ (g < 0)$$

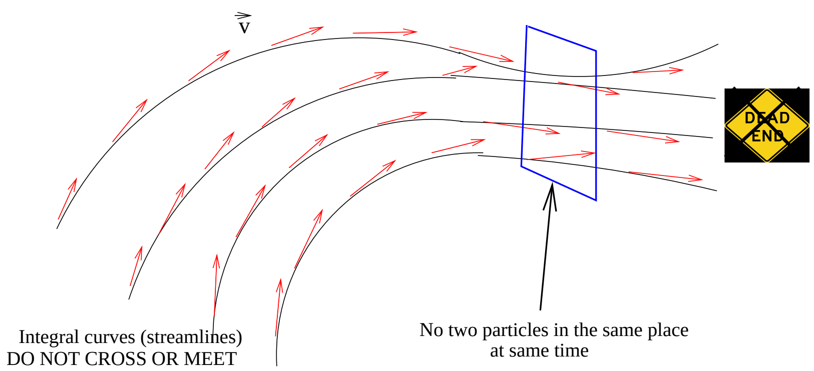
$$\frac{\hbar^2}{2m^2}\nabla^2\left(\frac{1}{\mathcal{R}}\nabla^2\mathcal{R}\right) \sim +\frac{1}{L^4} \;\leftarrow\; \textit{Repulsive at short distances}$$

Repulsion may prevent focussing/conjugate points. But it gets better!



No-crossing of quantal (Bohmian) trajectories

$$\vec{v}\,(\vec{x},t) = \frac{d\vec{x}}{dt} \equiv \frac{\hbar}{m}\,\mathcal{I}m\left(\frac{\vec{\nabla}\psi}{\psi}\right) = \frac{\hbar}{m}\,\vec{\nabla}S(\vec{x},t) \;= \text{single valued}$$



Trajectories do not end in a caustic/focus. They go on forever.

Relativistic generalization

$$\left[\partial^2 + \frac{m^2c^2}{\hbar^2} - \epsilon_1 R - \epsilon_2 \frac{i}{2} f_{cd}\sigma^{cd}\right]\Phi = 0$$

$(\Phi = \mathcal{R}e^{iS}$, Normalizable, single valued)

$$k_a = \partial_a S \ , \quad u_a = c\frac{dx_a}{d\lambda} = \frac{\hbar k_a}{m} \ , \quad \vec{v} = \frac{d\vec{x}}{dt} = -c^2\frac{\vec{\nabla}S}{\partial^0 S}$$

Imaginary part: $\partial^a\left(\mathcal{R}^2\partial_a S\right) = \frac{\epsilon_2}{2}\ f_{cd}\sigma^{cd}\mathcal{R}^2$

Real part: $k^2 = \frac{(mc)^2}{\hbar^2} - \epsilon_1 R + \frac{\partial^2\mathcal{R}}{\mathcal{R}}$

$$u^b{}_{;a}u^a = -\frac{\epsilon_1\hbar^2}{m^2}\ R^{;b} + \frac{\hbar^2}{m^2}\left(\frac{\partial^2\mathcal{R}}{\mathcal{R}}\right)^{;b}$$

(i.e. geodesic equation $+\ V_Q = \frac{\hbar^2}{m^2}\frac{\partial^2\mathcal{R}}{\mathcal{R}}$)

Quantum Raychaudhuri equation

$$\frac{d\theta}{d\lambda} = -\frac{1}{3}\ \theta^2 - \sigma_{ab}\sigma^{ab} - R_{cd}u^c u^d$$

$$-\frac{\epsilon_1\hbar^2}{m^2}h^{ab}R_{;a;b} - \frac{\hbar^2}{m^2}h^{ab}\left(\frac{\partial^2\mathcal{R}}{\mathcal{R}}\right)_{;a;b}$$

↑ *quantum corrections* ↑

Null geodesics: $\frac{d\theta}{d\lambda} = -\frac{1}{3}\ \theta^2 - \sigma_{ab}\sigma^{ab} - R_{cd}u^c u^d - \epsilon_1\hbar^2 h^{ab}R_{;a;b} - \hbar^2 h^{ab}\left(\frac{\partial^2\mathcal{R}}{\mathcal{R}}\right)_{;a;b}$

$\left[\theta = Tr(k_{a;b})\right]$

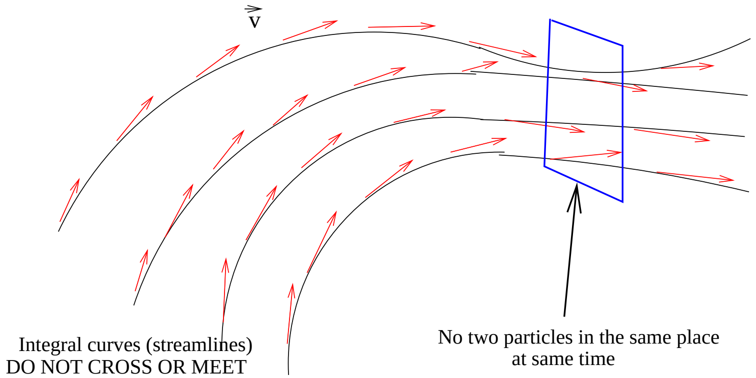
Again

$\Phi = \Phi_0 e^{-r^2/L^2}$, $\Phi = \Phi_0 \tanh(r/L\sqrt{2})$ $(g > 0)$, $\Phi = \sqrt{2}\,\Phi_0\ \mathrm{sech}(r/L)$ $(g < 0)$

$\frac{\hbar^2}{m^2}\left(\frac{\partial^2 \mathcal{R}}{\mathcal{R}}\right) \sim +\frac{1}{L^4}$ $\leftarrow$ *Repulsive at short distances. But, better...*

No-crossing of quantal (Bohmian) trajectories

$$\vec{v} = \frac{d\vec{x}}{dt} = -c^2 \frac{\vec{\nabla} S}{\partial^0 S}$$



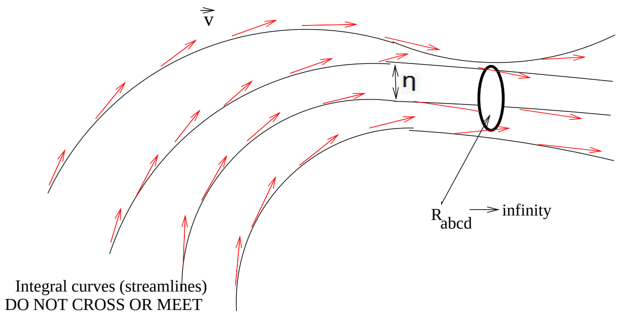
- No focusing, no conjugate points, geodesics go on forever
- No singularities! (all because of $\hbar$)

Curvature singularities

$$\frac{D^2\eta^a}{d\lambda^2} = -R^a{}_{bfc}u^b u^c \eta^f - \frac{\hbar^2}{m^2}\left[\left(\frac{\partial^2\mathcal{R}}{\mathcal{R}}\right)^{;a}\right]_{;c}\eta^c$$

$\uparrow$ *quantum*

But $\vec{\eta} \neq 0$ anymore



Therefore, $R_{abcd}R^{abcd} \to \infty$ regions are not accessible!

Application

Cosmology (Raychaudhuri $\rightarrow$ FRW)

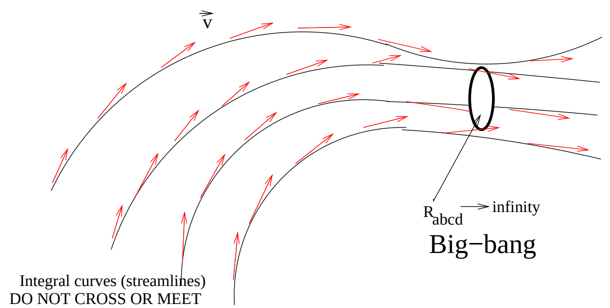
- Homogeneous, isotropic, $DE+DM+M$, GR cosmology, $a(t)=$ scale factor

$$\theta = 3\frac{\dot{a}}{a} \quad , \quad R_{cd}u^c u^d = \frac{4\pi G}{3}(\rho+3p) \quad \rightarrow$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho+3p) + \frac{\Lambda}{3} + \frac{\hbar^2}{3m^2} h^{ab}\left(\frac{\partial^2\mathcal{R}}{\mathcal{R}}\right)_{;a;b}$$

(second order Friedmann equation + quantum correction)

- No crossing (e.g. at the big-bang)



- Addition of (quantum) cosmological constant/dark energy

$$\Lambda_Q = \frac{\hbar^2}{m^2c^2} h^{ab}\left(\frac{\partial^2\mathcal{R}}{\mathcal{R}}\right)_{;a;b}$$

$\boxed{\Lambda_Q}$

For wavefunctions considered earlier

$(\psi = \psi_0 e^{-r^2/L^2}\,,\ \psi = \psi_0 \tanh(r/L\sqrt{2})\ (g>0)\ ,\ \psi = \sqrt{2}\ \psi_0\ \mathrm{sech}(r/L)\ (g<0))$

$$\Lambda_Q = \frac{1}{L^2} = \frac{1}{(\text{Compton wavelength})^2} = \left(\frac{mc}{h}\right)^2$$

$L = 1.4 \times 10^{26}\ m$ (Hubble radius/size of observable universe)

$m \approx 10^{-68}\ kg = 10^{-32}\ eV\ ,\quad F = -\frac{Gm_1m_2}{r^2}\,e^{-r/L}$

$$\Lambda_Q = 10^{-52} m^{-2} = 10^{-123} (\text{in Planck units}) = H_0^2/c^2$$

(Observed, also coincidence problem)

m = graviton (or photon) mass?

Consistent with observations and theory

S. Das, arXiv:1405.4011
A. F. Ali, S. Das, arXiv:1404.3093

Summary

- Classical $\rightarrow$ quantal (Bohmian) trajectories
  $\Rightarrow$ Classical $\rightarrow$ quantum Raychaudhuri equation
- No conjugate points, no geodesic incompleteness, no counterpart of singularity theorems for quantal trajectories
- Cosmology - addressing smallness and coincidence problems *and* no big bang
- Violation of the Equivalence principle? Quantum modifications of Einstein equations?
- Classical $\rightarrow$ *quantum* spacetime: results still expected to hold