Existential Elimination Made Easier

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The Bears Problem

- Consider the following sequent:

$$\exists x Fx, \forall x(Fx \rightarrow Gx) \vdash \exists x Gx.$$

- Instance in words: There are bears; all bears love honey; thus, there are lovers of honey. We'll call this the *Bears* argument.
- An argument as trivial as this ought to be technically easy to prove.

The "For Dummies" Method

1	(1)	$\exists x Fx$	A
2	(2)	$\forall x(Fx \rightarrow Gx)$	A
1	(3)	Fa	1 EE (?)
2	(4)	$Fa \rightarrow Ga$	2 UE
1,2	(5)	Ga	3,4 MP
1,2	(6)	$\exists x Gx$	5 EI

This is *invalid* since we don't know what a is, and so line (3) does not follow from line (1).

1	(1)	$\exists x Fx$	A
2	(2)	$\forall x(Fx \rightarrow Gx)$	A
3	(3)	Fa	A (EE)
2	(4)	$Fa \rightarrow Ga$	2 UE
2,3	(5)	Ga	3,4 MP
2,3	(6)	$\exists x Gx$	5 EI
1,2	(7)	$\exists x Gx$	1,3–6 EE

$\square$

Backing out of the subproof is an intricate process, and many students find it to be incomprehensible.

1	(1)	$\exists x Fx$	A
2	(2)	$\forall x(Fx \rightarrow Gx)$	A
	(3)	$a := xFx$	Df
1	(4)	Fa	1,3 EE
2	(5)	$Fa \rightarrow Ga$	2 UE
1,2	(6)	Ga	4,5 MP
1,2	(7)	$\exists x Gx$	6 EI

$\square$

How the Definition Works

- Consider $\exists x F x$.
- Can be read as "There is an F" or "There is an x such that Fx."
- $\exists$ is the verb or copula: "There is...".
- Then xFx is naturally read as "an F".
- This is the same as Hilbert's $\varepsilon x F x$, but for my purposes the ε is redundant.
  - So what I propose here could be called "an ε-calculus without the ε".
- This is easily extended to compound predicates; for instance, $x(Fx \wedge Gx)$ is "an F-ish G."
- I call an expression of the form xFx an *indefinite denotator* or *denotator* for short.

Using Definitions

- A definition is simply an agreement allowing the replacement of one string of symbols with another; it is a purely syntactic device which in effect is an expression in the metalanguage.
- We can define a rule DfIntro, which simply states that given a definition of one string in terms of another, either one can be replaced with the other in any line of a proof.

A Definition is not an Assumption

- Key simplification: We do not need to treat a definition such as $a := xFx$ as an *assumption*.
- It is perfectly in accord with usage in day-to-day and mathematical reasoning to introduce a new name for discussion purposes.
  - "Let Tony be the gunman on the grassy knoll. . ."
  - "Let p be a prime such that. . ."
- Exactly what working name or symbol we use makes *no* difference to the logic of the problem.
- It seems perfectly okay to allow judicious introduction of metalanguage into the object language of the proof.
- One could also think of definitions as being in the *subjunctive* mood (as in French), while the propositions in the object language of the proof are indicative.

Another Way of Thinking About Binding

- Normally, we think of free variables as bound by quantifiers:

$$\exists x F x$$

- It is more natural to think of variables as bound by predicate letters:

$$x F x.$$

Terms of Reference

- In this way of thinking about predicate calculus, there are three kinds of *terms*:
  - Proper names—attached to *one* entity.
  - Common names (aka arbitrary constants)—attached to an indefinite number of entities.
  - Denoting phrases (symbolized by denotators).
- An open sentence Fx can be turned into a proposition by replacing the free variable x with a proper name, a common name, or a denoting phrase.

Eliminating Quantifiers

- Hilbert argued that quantifiers can be defined in terms of ε-phrases, and thus eliminated.
- In denotator notation, the existential can be eliminated as follows:

$$\exists x F x \equiv F x F x \tag{HDR-Ex}$$

- "To say that there exists an F is to say that an F is F."
- Using these tools, we can do the Bears Problem without the use of names at all:

No-Names Method

1	(1)	$\exists x Fx$	A
2	(2)	$\forall x(Fx \rightarrow Gx)$	A
1	(3)	$FxFx$	1 HDR-Ex
2	(4)	$FxFx \rightarrow GxFx$	2 UE
1,2	(5)	$GxFx$	3,4 MP
1,2	(6)	$\exists x Gx$	5 EI

$\square$

"Look, ma, no names!"

Why the Working Names Method Works

- Hilbert's Denotator Rule also provides a derivation of the Working Names method of doing EE:

1	(1)	$\exists x F x$	A
1	(2)	$FxFx$	1 HDR-Ex
	(3)	$a := xFx$	Df
1	(4)	Fa	2,3 DfIntro

$\square$

The Susan Problem

- Here is another problem that can be simplified using denotators:

  > *Bob is looking at Susan. Susan is looking at Pablo. Bob is married, Pablo is not. Is a married person looking at an unmarried person?*

- Obviously, the answer is *yes*.
- How do we formalize the reasoning?
- It can be done ponderously with predicate calculus, but there is another way...

Tools to Solve the Susan Problem

- We'll use what I call the Combination Rule (CR):

$$Fa, Ga \vdash FxGx$$

- In words: given that a is both F and G, then a G is F.
- (One could, of course, also conclude that an F is G.)
    - (If Rufus is a bear, and Rufus is brown, then some bear is brown—or some brown thing is a bear.)

Formalizing the Susan Problem

Defining our names and predicates:

$$Lxy := x \text{ is looking at } y$$
$$Mx := x \text{ is married}$$
$$b := \text{Bob}; \quad s := \text{Susan}; \quad p := \text{Pablo}$$

Then the task is to establish the following sequent:

$$Lbs, Lsp, Mb, -Mp \vdash L(xMx)(y-My)$$

And Solving the Susan Problem

1	(1)	Lbs	A
2	(2)	Lsp	A
3	(3)	Mb	A
4	(4)	$-Mp$	A
	(5)	$Ms \vee -Ms$	ExM
6	(6)	Ms	A (vE)
2,6	(7)	$L(xMx)p$	2,6 CR
2,4,6	(8)	$L(xMx)(y-My)$	4,7 CR
9	(9)	$-Ms$	A (vE)
1,9	(10)	$Lb(y-My)$	1,9 CR
1,3,9	(11)	$L(xMx)(y-My)$	3,10 CR
1,2,3,4	(12)	$L(xMx)(y-My)$	5,6–8,9–11 vE

$\square$

Would We Always Want to Eliminate Names?

- Consider the following sequent:

$$\exists x \exists y Fxy, \forall x \forall y (Fxy \to Wx) \vdash \exists x Wx,$$

- A proof using the respectable method requires intricate, nested sub-proofs (even though this should be not much more difficult than the Bears Problem. . . !)
- It is easier to do using Working Names, as follows:

Working Names Proof of Double Quantifier Problem

1	(1)	$\exists x \exists y Fxy$	A
2	(2)	$\forall x \forall y (Fxy \rightarrow Wx)$	A
	(3)	$a := x \exists y Fxy$	Df.
1	(4)	$\exists y Fay$	1,3 EE
	(5)	$b := yFay$	Df.
1	(6)	Fab	4,5 EE
2	(7)	$\forall y (Fay \rightarrow Ga)$	2 UE
2	(8)	$Fab \rightarrow Ga$	7 UE
1,2	(9)	Ga	6,8 MP
1,2	(10)	$\exists x Gx$	9 EI

$\square$

Note how we eliminate the quantifiers one at a time, working from the outside in.

Can We Do This Proof Without Names?

- To do this using only denotators, we need to know how to eliminate the quantifiers from $\exists x \exists y Fxy$.
- Do one variable at a time, treating the other as if it were a constant (analogously to partial differentiation).
- Eliminate the quantifier from $\exists x Fax$, where a is a constant:

$$\exists x Fax \equiv Fa(xFax)$$

and

$$\exists x Fxa \equiv F(xFxa)a.$$

(So in effect we are treating Fa as if it were the predicate in the Denotator Rule for Existentials.)

Now Try the Double Quantifier Problem. . .

- Now break down the doubly-quantified statement in the problem, starting with the inner quantifier and using the previous slide:

$$\exists x \exists y Fxy \equiv \exists x(Fx(yFxy)) \tag{1}$$
$$\equiv F(xFx(yFxy))(yFxy)) \tag{2}$$

- I won't go any further, because this is so hopelessly complicated that it offers no practical computational advantage over the Working Names method.
- What may be conceptually simpler is not always notationally simpler!

What's In A Name?

- So what is in a name?
- For one thing, *concision*!
- In some very simple problems, such as the Bears problem, or in problems expressed in terms of denoting phrases, as in the Susan problem, it may be useful to skip the use of names.
- However, my guess is that the Working Names method is in general the simplest practical method of doing Existential Elimination—unless some entirely different way of symbolizing quantificational reasoning can be devised.

Denotators and Definite Descriptions

- Simplifications can be made in solving the usual textbook problems about definite descriptions.
- $$\begin{array}{l} \textit{Rudolph is the lead reindeer on Santa's sleigh.} \\ \textit{Blitzen is not Rudolph.} \\ \therefore \textit{Blitzen is not the lead reindeer on Santa's sleigh.} \end{array}$$
- This is intuitively valid. How can we prove it in a way that is consistent with its triviality?

Symbolic Tools for Definite Denotations

- There are three symbolic tricks we can use:
  - Treat "the F" as "an unique F".
  - Then "the F" can be symbolized as a denotator of the form

    $$x(Fx \wedge \forall y(Fy \rightarrow (y = x))).$$

    (The second clause, expressing uniqueness, is of course borrowed from Russell.)
  - Borrow the Russell inverted iota to express this more compactly:

    $$\imath xFx := x(Fx \wedge \forall y(Fy \rightarrow x = y)).$$

- So xFx is "an F", and $\imath xFx$ is "the F".

And a Deductive Tool

- Since the referent of a definite description (if it exists) is *unique*, such objects can stand in identity relations.
- Thus, we can use the standard rules for identity introduction and elimination in reasoning about such objects.

An Example

- E.g.:

$$\begin{array}{l} \textit{Trudeau is the Prime Minister of Canada.} \\ \textit{The Prime Minister of Canada is a Quebecer.} \\ \therefore \textit{ Trudeau is a Quebecer.} \end{array}$$

- Proof:

1	(1)	$t = \imath x Px$	A
2	(2)	$Q\imath x Px$	A
1,2	(3)	Qt	1,2 =E

□

Back to Rudolph...

- Proof of the Rudolph problem:

1	(1)	$r = \imath x L x$	A
2	(2)	$b \neq r$	A
1,2	(3)	$b \neq \imath x L x$	1,2 EE

$\square$

- What makes it so simple is that we do not have to take apart the denoting phrase in order to prise out the consequences of the uniqueness clause.

One More Example...

- Here is another example that ought to be easy to prove:

$$\begin{array}{l} \textit{Shakespeare was the author of Hamlet.} \\ \textit{The author of Hamlet was the author of Macbeth.} \\ \therefore \textit{Shakespeare was the author of Macbeth.} \end{array}$$

- Defining our symbols:

$$\begin{array}{l} \mathsf{Axy} := \mathsf{x} \textit{ is an author of } \mathsf{y} \\ \mathsf{h} := \textit{Hamlet} \\ \mathsf{m} := \textit{Macbeth} \\ \mathsf{s} := \textit{Shakespeare.} \end{array}$$

Solving the Shakespeare Problem

- We can define working *proper* names to facilitate a proof like this:

	(1)	$a := \imath x(Axh)$	Df.
	(2)	$b := \imath x(Axm)$	Df.
3	(3)	$a = b$	A
4	(4)	$s = a$	A
3,4	(5)	$s = b$	3,4 $=$E

- Not all definite description problems can be handled quite so neatly, but in this case, at least, the simplicity of the derivation matches the triviality of the deduction.

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