The Square Root of Falsehood *for* World Logic Day, 2020.

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January 14, 2020

Abstract

I will explain some new approaches to long-standing puzzles in classical logic.

The Paradox that Keeps on Giving

- When we teach "baby" propositional logic we pretend that every proposition has a definite truth value, T or F.
  - I will begin by showing you that this is false!
- It is easy to construct sentences which either cannot be consistently valuated, or can be consistently valuated in more than one way.

The Liar

- Consider this:

$$\text{(1) is false.} \tag{1}$$

- Says of itself that it is false.
  - Thus, if it is true, it is false!
  - However, if it is false, then what it says must be true!
- Impossible to consistently valuate this sentence, which has been debated since ancient times.
- Should we simply declare such self-referential claims illegitimate, ill-formed?

The Truth-Teller

- Now try this:

  (2) is true. (2)

- Says of itself that it is true.
  - Unlike the Liar, it is consistent, since if it is true, it is true.
  - However, if it is false, it is false!
- Either picture is consistent, but we have to arbitrarily decide whether it should be considered true or false; there is nothing to go on.

Groundedness

- Consider:
  - There was a gunman on the grassy knoll.
  - $2^{82,589,933} - 1$ is prime.
- The truth value of the first statement is unknown; the second is true.
- Both are *grounded* in something outside language; the first in facts of history, the second in math.
- Self-referential statements such as the Liar and the Truth-Teller are said to be *ungrounded*.

Curry's Paradox

- This is a bit more complicated:

$$\text{If (3), then } p. \tag{3}$$

- We can look at this a couple of ways.
  - If we insist that (3) has a consistent truth value, then there is no way for (3) to be F, because (by the truth table for "if–then–"), (3) would have to be T; contradiction!
  - Hence, if (3) is definitely T or F, and it can't be F, it must be T; and then by the truth table p must be T as well—even though p could be any proposition whatsoever!
  - (Basis of familiar "Knights and Knaves" paradoxes by Raymond Smullyan.)

Curry's Paradox

- *Or...*
  - If we know p is T, then (3) must be T, and this is consistent.
  - If we know p is F, then if (3) is F it is T; contradiction!
  - Thus, if we know p is F, then if (3) is T it is F; contradiction!
- So value of p is like a control parameter that determines whether or not (3) has a consistent valuation, or oscillates like the Liar.
  - So—how should we think about these paradoxes?

Infinite Regress?

- Medieval logicians might have rejected the Liar as well-formed since from one viewpoint it leads to an infinite regress (if it's true, it's false; if it's false, it's true. . .).
- Thinkers such as Aquinas thought that infinite regresses were absurd; they did not know about infinite series.
  - This was the basis for one of Aquinas's "proofs" for ∃God: there must be a first cause, since an infinite regress of causes would be absurd. Simply a circular argument from the modern point of view.
- There seems to be no good reason why we should rule out an infinite regress.
  - Example: solve the following equation for x:

$$x^{x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}} = 2.$$

  - Obviously, this is the same thing as

$$x^2 = 2$$

Herzberger's Naïve Semantics

- Powerful suggestion was made by Hans Herzberger [1], building on concepts from Tarski and Kripke.
  - It is impossible to make general rules to prevent the formulation of paradoxical statements such as the Liar or Curry's Paradox.
  - Let us simply accept their behaviour as a natural feature of language and mathematical logic.
  - Only grounded sentences can be guaranteed to have stable truth values.
  - We go beyond the truth value: self-referential forms such as the Liar have *truth-patterns* or *wave-forms*, which could be perfectly sensible objects of mathematical study.

Russell's Recondite Property

- Russell [2] pointed out that paradoxical statements have the "recondite" property of self-reference or implicitness.
- We can write the Liar as

$$p = -p \tag{4}$$

- This is said to be *implicit*.
  - Compare with explicit form such as

    $$p = q \vee r.$$

    Easy to solve; just substitute in the truth values for q and r, and use the truth table for $\vee$.
- But what do we do with (4)?

Beyond the Equation

- Maybe we need to move beyond equations, and work with *recursion relations*.
- Write the Liar as follows:

$$p_0 = \mathsf{T} \tag{5}$$

$$p_{n+1} = -p_n. \tag{6}$$

- Then we can easily compute Herzberger's truth pattern, step by step.
- The choice of initial value for p_0 is arbitrary; I could have set it to F.
- That would merely change the *phase* of the output square wave.

Complex Truth Values?

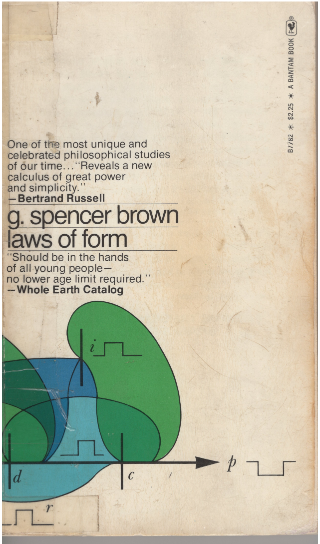
- George Spencer Brown [3] pointed out analogy between Liar and algebraic equations that have no solution *in real numbers*.

$$x^2 = -1$$

- We can define an "imaginary" root

$$i = \sqrt{-1}$$

and then "solve" the equation with $x = \pm i$.



Is there a Fundamental Theorem of Logic?

- More generally, we can define *complex* numbers of the form $a + ib$, where a and b are reals.
- These have *huge* utility in pure and applied mathematics, even though their intuitive meaning is not immediately obvious.
  - Sometimes we have to let formalism guide our intuitions!
- Many algebraic equations have no real-valued solutions.
- However, Gauss and others proved the following very powerful theorem, called The Fundamental Theorem of Algebra [4]:

  Every algebraic equation of degree n has precisely n roots, real or complex.

- Could something similar hold for logic? Could there be *complex truth values*, based on $\sqrt{\text{NOT}}$?

Quantum Computation

- In fact, there is!
- In quantum computation [5], it is possible to define a unitary operator that behaves very much like what we would need in order to carry out Herzberger's "naïve" semantics:

$$\sqrt{\mathrm{NOT}} = \frac{1-i}{2}\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} = \frac{1+i}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \tag{7}$$

- When multiplied with itself, it gives another operator that inverts a "proposition" (state vector).
- Could we apply tools of quantum computation (together with Herzberger's viewpoint) to "solve" the Liar?
- Work in progress!

References I

[1] Herzberger, Hans. “Notes on Naïve Semantics,” *Journal of Philosophical Logic* 11(1982), 61–102.

[2] Russell, Bertrand *Essays in Analysis*, Douglas Lackey (ed.), New York: George Braziller.

[3] Brown, George Spencer. *The Laws of Form*. London: Allen & Unwin, 1969. (Bantam edition, 1973.)

[4] “The Fundamental Theorem of Algebra,” *Wikipedia* https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra

[5] Lomonaco, Samuel L. Jr. “A Rosetta Stone for Quantum Mechanics With an Introduction to Quantum Computation 1.5”. arXiv:quant-ph/0007045v1