# Some recent advances in Quantum Gravity Phenomenology

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*Black Holes IX*
Saskatoon, 15 May 2013

Plan:

- Why Quantum Gravity Phenomenology?
- Recent Progress (1994, 1999, 2008-...)
- Robustness
- Results
- Mathematical Aspects
- Proposed Experiments
- Summary and Outlook

## A problem with Quantum Gravity

- Too many theories: *String Theory, Loop Quantum Gravity, Non-Commutative Field Theory, Dynamical Triangulations, Causal Sets,...*
- Too few experiments $= Zero$
- Why? *Quantum Gravity effects expected at the Planck Scale* $\approx 10^{16}\ TeV$

  Atomic Physics $\approx 10\ eV \ \approx 10^{-11}\ TeV$ . LHC $\approx 10\ TeV$
- Difference of $15 - 27$ orders of magnitude

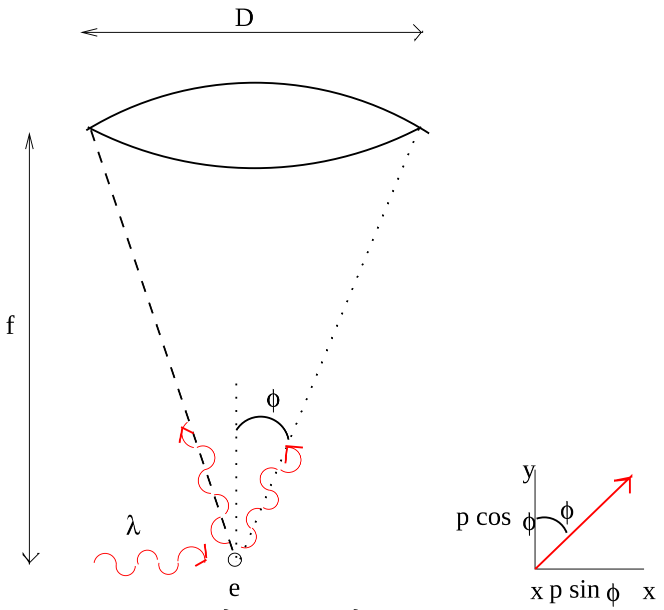
$$\boxed{\text{QG} \rightarrow \text{Experimental Signatures?}}$$

## Recent Progress

- Generalized/Modified Uncertainty Principle (GUP)
- Black Hole Physics
- String Theory
- Loop Quantum Gravity, via Polymer Quantization
- Non-commutative Geometry
- ······

**A meeting ground for various theories?**

## Heisenberg's Microscope



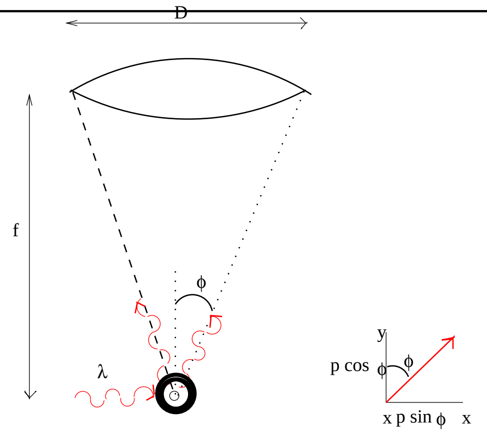

Position uncertainty $\Delta x \geq f\frac{\lambda}{D} \approx \frac{\lambda}{2\phi}$ $\left(\textit{Minimum Resolving Power}\right)$

Momentum uncertainty $\Delta p = p \sin\phi = \frac{\hbar}{\lambda}\sin\phi \approx \frac{\hbar\phi}{\lambda}$

$$\boxed{\Delta p \Delta x \geq \frac{\hbar}{2}}$$

*Heisenberg Uncertainty Principle*

## Heisenberg's Microscope with a Black Hole (Extremal RN)




$$r_+ = GM + \sqrt{(GM)^2 - GQ^2}$$

$$\Delta x_{new} = r_+(M + \Delta M) - r_+(M)$$

$$= G\Delta M + \sqrt{(GM + G\Delta M)^2 - GQ^2} - \sqrt{(GM)^2 - GQ^2} \geq 2G\Delta M \sim \frac{\ell_{Pl}^2}{\lambda} (\Delta M = \frac{\hbar}{\lambda}, \Delta p = \frac{\hbar \sin\phi}{\lambda})$$

$$\sim \frac{\ell_{Pl}^2 \Delta p}{\hbar \sin\phi} \geq \frac{\ell_{Pl}^2 \Delta p}{\hbar} \to \Delta x + \Delta x_{new} \geq \frac{\hbar}{\Delta x} + \beta_0 \frac{\ell_{Pl}^2 \Delta p}{\hbar}$$

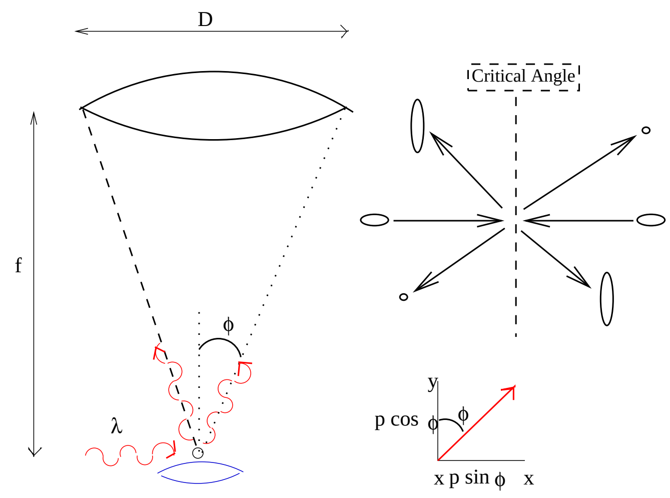
$$\boxed{\Delta p \Delta x \geq \frac{\hbar}{2}\left[1 + \beta_0 \frac{\ell_{Pl}^2}{\hbar^2}\Delta p^2\right]} \leftarrow (M, Q) \text{ independent}$$

*Generalized Uncertainty Principle*

$\ell_{Pl} = \sqrt{\frac{G\hbar}{c^3}} = 10^{-35} m$. The new term is effective only when $x \approx \ell_{Pl}$ or $p \approx 10^{16} TeV/c$

M. Maggiore, Phys. Lett. **B304**, 65 (1993)

## Heisenberg's Microscope with an elementary string




$$\Delta x_{new} \sim \sqrt{E} \sim p \geq \frac{\ell_{Pl}}{\hbar} \Delta p$$

$$\boxed{\Delta p \Delta x \geq \frac{\hbar}{2} \left[ 1 + \beta_0 \frac{\ell_{Pl}^2}{\hbar^2} \Delta p^2 \right]}$$

*Generalized Uncertainty Principle*

D. Amati, M. Ciafaloni, G. Veneziano, Phys. Lett. **B216** 41 (1989)

## Minimum Observable Length from the GUP

$$\Delta p \Delta x \geq \frac{\hbar}{2}\left[1 + \beta_0 \frac{\ell_{Pl}^2}{\hbar^2} \Delta p^2\right]$$

$$\text{Invert} \quad \Delta p \leq \frac{\hbar}{\ell_{Pl}}\left[1 \pm \sqrt{\Delta x^2 - \beta_0 \ell_{Pl}^2}\right]$$

$$\boxed{\Delta x \geq \sqrt{\beta_0}\,\ell_{Pl} \equiv \Delta x_{min}}$$

(Min length)

$\boxed{\text{Note: } \Delta x_{min} \not\Rightarrow \text{Space is discrete}}$

.

$$\Delta p \Delta x \geq | < [x,p] > |$$

| | HUP | GUP |
|---|---|---|
| Principle | $\Delta p_i \Delta x_i \geq \frac{\hbar}{2}$ | $\Delta p_i \Delta x_i \geq \frac{\hbar}{2}\left[1 + \beta_0 \frac{\ell_{Pl}^2}{\hbar^2} \Delta p^2\right]$ |
| Algebra | $[x_i, p_j] = i\hbar\delta_{ij}$ | $[x_i, p_j] = i\hbar[\delta_{ij} + \underbrace{\frac{\beta_0}{\hbar^2}\ell_{Pl}^2(p^2\delta_{ij} + 2p_ip_j)}_{New}]$ |

A. Kempf, G. Mangano, R. B. Mann, Phys. Rev. **D52** 1108 (1995)

## Problems

- $10^{16} Tev$ , $10^{-35}\ m$ (Planck scale/QG) in 'whose frame'??
  $\rightarrow$ Problem of *Lorentz Covariance of QG*
- Effects $\propto \ell_{Pl}^2 \approx 10^{-70} m^2 \quad \leftarrow$ *Bad for phenomenology*

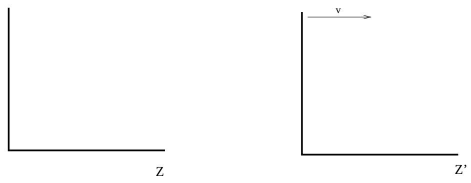
$$\boxed{\textit{Is there a GUP linear in } \ell_{Pl}\textit{?}}$$

- Postulate a linear term (Reasonable. May put bounds on QG parameters)
- Look for a linear term

One motivation: new $[x,p]$ algebra from Doubly Special Relativity Theories

(One way to solve the 'QG in whose frame' problem)

$$[J_i, K_j] = \epsilon^{ijk} K_k \ , \ [K^i, K^j] = \epsilon^{ijk} J_k \text{ But } K^i = L_0^i + \ell_{Pl} p^i \ p_a \frac{\partial}{\partial p_a}$$




$$\begin{aligned}
p_0' &= \frac{\gamma(p_0 - v p_z)}{1 + \ell_{Pl}(\gamma - 1)p_0 - \ell_{Pl}\gamma v p_z} \\
p_z' &= \frac{\gamma(p_z - v p_0)}{1 + \ell_{Pl}(\gamma - 1)p_0 - \ell_{Pl}\gamma v p_z} \\
p_x' &= \frac{\gamma(p_z - v p_0)}{1 + \ell_{Pl}(\gamma - 1)p_0 - \ell_{Pl}\gamma v p_z} \\
p_y' &= \frac{\gamma(p_z - v p_0)}{1 + \ell_{Pl}(\gamma - 1)p_0 - \ell_{Pl}\gamma v p_z}
\end{aligned}$$

Minimum (invariant) length: $\ell_{Pl}$

Maximum (invariant) momentum: $p_{0Max} = M_{Pl} c = \frac{\hbar}{\ell_{Pl}}$

$$E^2 - \vec{p}^2 \neq m^2$$

$$E^2 f(E, \vec{p}, \ell_{Pl}) - \vec{p}^2 g(E, \vec{p}, \ell_{Pl}) = m^2 \equiv \epsilon^2 - \vec{\pi}^2$$

$$[x_i, p_j] = i\hbar \frac{\partial p_i}{\partial \pi_j}$$

$$f = \frac{|\vec{p}|}{E} \ , \ g = \frac{1}{1 - \ell_{Pl}|\vec{p}|} \ \text{(massless)}$$

$$[x_i, p_j] = i\hbar[(1 - \ell_{Pl}|\vec{p}|)\delta_{ij} + \ell_{Pl}^2 p_i p_j]$$


J. Magueijo, L. Smolin, Phys. Rev. Lett. **88** 190403 (2002),

J. L. Cortes, J. Gamboa, Phys. Rev. **D71** 026010 (2005)

So now we have *two* new algebras/GUPs

Most general quadratic in $p$ algebra: use Jacobi identity

$$-\Big[[x_i, x_j], p_k\Big] = \Big[[x_j, p_k], x_i\Big] + \Big[[p_k, x_i], x_j\Big] = 0$$

$$\boxed{[x_i, p_j] = i\hbar\left[\delta_{ij} - \alpha\left(p\delta_{ij} + \frac{p_i p_j}{p}\right) + \alpha^2(p^2\delta_{ij} + 3p_i p_j)\right]}$$

$$\Delta x \geq (\Delta x)_{min} \approx \alpha_0 \ell_{Pl} \ , \ \Delta p \leq (\Delta p)_{max} \approx \frac{M_{Pl}c}{\alpha_0}$$

$$\alpha = \frac{\alpha_0}{M_{Pl}c} = \frac{\alpha_0 \ell_{Pl}}{\hbar} \ . \ \alpha_0 = \mathcal{O}(1) \text{ (normally)}$$

Although current experiments $\Rightarrow \alpha_0 \leq 10^{17} \rightarrow \quad \alpha^{-1} \approx 10\ TeV/c$

## Other approaches

- Non-commutative geometry
- Polymer quantization


A. Kempf, J. Math. Phys. **35** (1994) 4483-4496 (arXiv:9311147)

S. Pramanik, S. Ghosh, arXiv:1301.4042

V. Husain, S. S. Seahra, E. J. Webster, arXiv:1305.2814

## Consequences

$[x_i, p_j] = i\hbar \left[ \delta_{ij} - \alpha \left( p\delta_{ij} + \frac{p_i p_j}{p} \right) + \alpha^2 \left( p^2 \delta_{ij} + 3 p_i p_j \right) \right] \Rightarrow \quad p_j \neq -i\hbar \frac{\partial}{\partial x_i}$ ~~pos'n space~~

But define:

$$\boxed{p_j = p_{0j} \left(1 - \alpha p_0 + 2\alpha^2 p_0^2\right)}$$ with $[x_i, p_{0j}] = i\hbar\delta_{ij}$ , $p_{0j} = -i\hbar \frac{\partial}{\partial x_j}$

$[x_i, p_j] = \ldots$ is satisfied

## Consider any Hamiltonian

$$H = \frac{p^2}{2m} + V(\vec{r}) = \frac{1}{2m} \left( p_{0j} \left( 1 - \alpha p_0 + 2\alpha^2 p_0^2 \right) \right)^2 + V(r)$$

$$= \underbrace{\frac{p_0^2}{2m} + V(\vec{r})}_{H_0} \underbrace{- \frac{\alpha}{m} \, p_0^3 + \mathcal{O}(\alpha^2)}_{H_1} = \frac{p_0^2}{2m} + V(\vec{r}) \quad \underbrace{- \frac{i\hbar^3 \alpha}{m} \frac{d^3}{dx^3}}_{\text{position space}}$$

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**Quantum gravity effect in any quantum system!**

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## Schrödinger Equation

$$[H_0 + H_1]\psi = \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) - i\frac{\alpha\hbar^3}{m}\frac{d^3}{dx^3}\right]\psi = i\hbar\frac{\partial\psi}{\partial t}$$

*Two Consequences*

**1. New Perturbed Solutions *and* New Conserved Current**

$$J = \frac{\hbar}{2mi}\left(\psi^\star\frac{d\psi}{dx} - \psi\frac{d\psi^\star}{dx}\right) + \frac{\alpha\hbar^2}{m}\left(\frac{d^2|\psi|^2}{dx^2} - 3\frac{d\psi}{dx}\frac{d\psi^\star}{dx}\right)$$

$\rho = |\psi|^2$ , $\frac{\partial J}{\partial x} + \frac{\partial \rho}{\partial t} = 0 \rightarrow$ *New Reflection/Transmission Currents*

**2. New *Non-Perturbative* solution $\sim e^{ix/\ell_{Pl}}$** $\rightarrow$ *Discreteness of space*

*Also - Dirac equation...*

## Applications to Quantum Mechanical Systems

- Simple Harmonic Oscillator
- Landau Levels
- Lamb Shift
- Scanning Tunneling Microscope
- Particle in a box
- Superconductivity
- Quantum Hall Effect
- $(g-2)$ of muon
- Black Holes in LHC (A. F. Ali)

$\boxed{\text{Precision required to test GUP: 1 part in } 10^{12} - 10^{25}}$

S. Das, E. C. Vagenas, Phys. Rev. Lett. **101**, 221301 (2008), arXiv:0810.5333
A. F. Ali, S. Das, E. C. Vagenas, Phys. Rev. **D84** (2011) 044013

S. Das, R. B. Mann, Phys. Lett. **B704** (2011) 596-599, arXiv:1109.3258

## Simple Harmonic Oscillator)

$$H = \underbrace{\frac{p_0^2}{2m} + \frac{1}{2}m\omega x^2}_{H_0} \quad \underbrace{+\frac{\alpha}{m}p^3 + \frac{3\alpha^2}{2}p^4}_{H_1}$$

$$\left[\psi_n = \frac{1}{2^n n!}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{\frac{-m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\right]$$

$$\Delta E_{GUP} = \langle\psi_n|H_1|\psi_n\rangle|_{(p^4\ \text{term})} + \sum_{k\neq n}\frac{|<k^0|H_1|n^0>|^2}{E_n^0 - E_k^0}|_{(p^3\ \text{term})}$$

$$\frac{\Delta E_{GUP(0)}}{E_0} = \frac{9}{2}\hbar^2\omega^2 m\alpha^2 + 4\hbar\omega m\alpha^2$$

$$\boxed{\Delta E_{GUP(0)} \sim \alpha^2}$$

# Landau Levels

Particle of mass $m$, charge $e$ in constant $\vec{B} = B\hat{z}$, i.e.$\vec{A} = Bx\hat{y}$, $\omega_c = eB/m$

$$H_0 = \frac{1}{2m}\left(\vec{p}_0 - e\vec{A}\right)^2 = \frac{p_{0x}^2}{2m} + \underbrace{\frac{p_{0y}^2}{2m}}_{\hbar^2 k^2/2m} - \frac{eB}{m}\, x p_{0y} + \frac{e^2 B^2}{2m}\, x^2 = \frac{p_{0x}^2}{2m} + \frac{1}{2} m\omega_c^2 \left(x - \frac{\hbar k}{m\omega_c}\right)^2$$

$$H = \frac{1}{2m}\left(\vec{p}_0 - e\vec{A}\right)^2 - \frac{\alpha}{m}\left(\vec{p}_0 - e\vec{A}\right)^3 = H_0 - \sqrt{8m}\,\alpha H_0^{\frac{3}{2}}$$

$$\frac{\Delta E_{n(GUP)}}{E_n} = -\sqrt{8m}\alpha(\hbar\omega_c)^{\frac{1}{2}}(n + \tfrac{1}{2})^{\frac{1}{2}} \approx -10^{-27}\alpha \quad (B = 10\ T)$$

## Conclude

- $\alpha \sim 1$ and $\frac{\Delta E_{n(GUP)}}{E_n}$ is too small, *or*
- Measurement accuracy of 1 in $10^3$ in STM $\rightarrow$ $\alpha_0 < 10^{24}$

## Lamb Shift

$$H_0 = \frac{p_0^2}{2m} - \frac{k}{r} \ , H_1 = -\frac{\alpha}{m} p_0^3 = (\alpha\sqrt{8m}) \left[H_0 + \frac{k}{r}\right] \left[H_0 + \frac{k}{r}\right]^{\frac{1}{2}}$$

$$\Delta E_n = \frac{4\alpha^2}{3m^2} \left(\ln\frac{1}{\alpha}\right) |\psi_{nlm}(0)|^2 \ \ (\textit{Lamb Shift})$$

$$\frac{\Delta E_{n(GUP)}}{\Delta E_n} = 2\frac{\Delta|\psi_{nlm}(0)|}{\psi_{nlm}(0)}$$

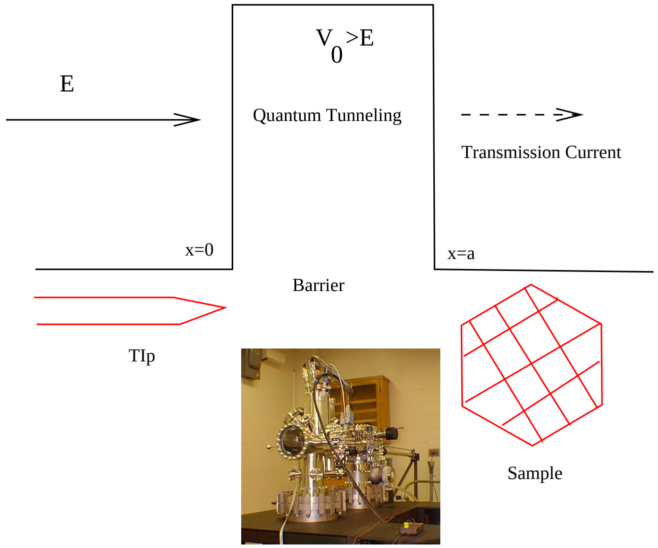
$$\Delta\psi_{100}(\vec{r}) = \sum_{\{n'l'm'\}\neq\{nlm\}} \frac{\langle n'm'l'|H_1|nlm\rangle}{E_n - E_{n'}} \langle\vec{r}|n'l'm'\rangle = 10^3 \alpha E_0 \ \psi_{200}(\vec{r})$$

$$\frac{\Delta E_{n(GUP)}}{\Delta E_n} = 2\frac{\Delta|\psi_{nlm(0)}|}{\psi_{nlm}(0)} \approx \alpha_0 \frac{4.2\times 10^4 E_0}{27 M_{Pl} c^2} \approx 10^{-24}\alpha_0$$

## Conclude

- $\alpha \sim 1$ and $\frac{\Delta E_{n(GUP)}}{E_n}$ is too small, *or*
- Measurement accuracy of 1 in $10^{12}$ $\rightarrow$ $\alpha_0 < 10^{12}$ (*better!*)

# Potential Barrier (Scanning Tunneling Microscope)




$$[H_0 + H_1]\psi = E\psi \qquad [H_0 + H_1]\psi = -(V_0 - E)\psi \qquad [H_0 + H_1]\psi = E\psi$$

$$\psi_1 = Ae^{ik'x} + Be^{-ik''x} + Pe^{\frac{ix}{2\alpha\hbar}} \qquad \psi_2 = Fe^{k'_1 x} + Ge^{-k''_1 x} + Qe^{\frac{ix}{2\alpha\hbar}} \qquad \psi_3 = Ce^{ik'x} + Re^{\frac{ix}{2\alpha\hbar}}$$

$$k = \sqrt{\frac{2mE}{\hbar^2}}\ ,\ \ k_1 = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$$

$$k' = k(1 + \alpha\hbar k),\ k'' = k(1 - \alpha\hbar k),\ k'_1 = k_1(1 - i\alpha\hbar k_1),\ k''_1 = k_1(1 + i\alpha\hbar k_1)$$

Take into account

- Continuity of $\psi, \psi', \psi''$ at each boundary (cannot set $P, Q, R = 0$)
- New current

Transmission Current

$$T = \frac{J_R}{J_L} = \left|\frac{C}{A}\right|^2 - 2\alpha\hbar k \left|\frac{B}{A}\right|^2 .$$

$$= T_0\left[1 + 2\alpha\hbar k(1 - T_0^{-1})\right], \quad T_0 = \frac{16E(V_0 - E)}{V_0^2} e^{-2k_1 a} = \text{usual}$$

$$m = m_e = 0.5\ MeV/c^2\ ,\ E \approx V_0 = 10\ eVa = 10^{-10}\ m\ ,\ I = 10^{-9}\ A\ ,\ \mathcal{G} = 10^9\ ,$$

$$I \propto T$$

$$\frac{\delta I_{GUP}}{I_0} = \frac{\delta T_{GUP}}{T_0} = 10^{-26},$$

$$\delta\mathcal{I}_{\mathcal{GUP}} = \mathcal{G}\delta I_{GUP} = 10^{-26}A, \alpha_0 = 1, T_0 = 10^{-3}$$

$$\boxed{\tau = \frac{e}{\delta\mathcal{I}_{\mathcal{GUP}}} = 10^5 - 10^7\ s \approx \text{ a day - a month}}$$

# Superconductivity

$$\vec{J} = \frac{\hbar}{2mi}\left[\psi^\star\vec{\nabla}\psi - \psi\vec{\nabla}\psi^\star\right] + \frac{5a^2\hbar^3 e}{2mi}\left[\left(\psi^\star\vec{\nabla}\nabla^2\psi - \psi\vec{\nabla}\nabla^2\psi^\star\right) + \left(\nabla^2\psi^\star\vec{\nabla}\psi - \nabla^2\psi\vec{\nabla}\psi^\star\right)\right]$$

$$\equiv \vec{J}_0 + \vec{J}_1, \quad Charge = -2e\ , \quad Mass = 2m\ , \quad \psi = |\psi|e^{i\phi}\ , \quad \vec{\nabla}|\psi| = 0$$

$$\vec{J}_0 = -\left[\frac{2e^2}{mc}\vec{A} + \frac{e\hbar}{m}\vec{\nabla}\phi\ .\right]|\psi|^2 \text{ (usual SC current)}$$

$$J_1 = -\frac{80a^2e^4}{mc^3}\vec{A}|\vec{A}|^2|\psi|^2 \text{ (QG effect)}$$

$$\oint \vec{J}\cdot d\vec{l} = \oint \vec{J}_0\cdot d\vec{l} + \oint \vec{J}_1\cdot d\vec{l} = 0$$

$$\Phi = n\left(\Phi_0 + \alpha^2\Phi_1\right)$$

$$\Phi_0 = \frac{hc}{2e}\ , \quad \Phi_1 = \frac{40e^2|\vec{B}|^2L^2}{c^2}\ \Phi_0 \quad \text{(Flux quantization)}$$

$\alpha_0 < \frac{10^{-n/2}}{\sqrt{40}}\frac{M_{Pl}c^2}{eBL} < 10^{19-n/2}$ (experimental precision of 1 part in $10^n$)

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S. Das, R. B. Mann, Phys. Lett. **B704** (2011) 596-599, arXiv:1109.3258

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# Quantum Hall Effect

$e\mathcal{E}_y = evB$

$\mathcal{E}_y = \frac{j_x B}{ne}$

$\mathcal{E}_y = \rho_{xy} j_x$

$\rho_{xy} = \frac{B}{ne}$

Landau levels $E_n = \hbar\omega_c \left(n + \frac{1}{2}\right)$ $\omega_c = eB/mc =$ cyclotron frequency

$n_H = \frac{B}{\Phi_0 + \alpha^2 \Phi_1}$

Fermi energy $E_F$ lies between the energy levels $E_k$ and $E_{k+1}$, all states $E_i$ , $i \le k$ are occupied, resulting in the carrier density $n = kn_H$

$$\rho_{xy} = \frac{hc}{ke^2}\left[1 + \frac{10\ \alpha^2\ e^2|\vec{B}|^2 L^2}{c^2}\right]$$

## Anomalous magnetic moment of the muon: Dirac equation

$$i\hbar \frac{\partial \chi_1}{\partial t} = \Big[ \Big( \frac{1}{2m} - c\alpha \Big) |\vec{\Pi}|^2 - \frac{\alpha}{(2m)^2 c} \Pi^4 + e\phi \chi_1$$

$$-2 \frac{e\hbar}{2mc} \Big( 1 - 2\alpha c m - \frac{\alpha}{mc} \Pi^2 \Big) \vec{S} \cdot \vec{B} - \frac{\alpha e^2 \hbar^2}{(2m)^2 c^3} |\vec{B}|^2$$

$$- \frac{ie\hbar\alpha}{2(mc)^2} \Big( \vec{\nabla}(\vec{\sigma} \cdot \vec{B}) \cdot \vec{\Pi} - \frac{e}{c} \vec{A} \cdot \vec{\nabla}(\vec{\sigma} \cdot \vec{B}) + i\hbar \nabla^2 (\vec{\sigma} \cdot \vec{B}) \Big) \Big] \chi_1$$

$$g = 2(1 - 2acm - \frac{\alpha}{mc} \Pi^2)$$

$$\left( \frac{g-2}{2} \right)_{GUP} = - \left[ 2\alpha c m + \frac{\alpha \Pi^2}{mc} \right] .$$

$$\alpha_0 < 10^{-n} \frac{m_{Pl}}{m_\mu} < 10^{20-n} \ ,$$

GUP effects on

Atomic/Molecular/Condensed Matter Systems

*Stark Effect, Zeeman Effect, Berry's Phase, Bohm-Aharonov effect, Dirac Quantization, Anomalous Magnetic Moment of Electron, Anderson Localization, Coherent States, Lasers,...*

Statistical Mechanical Systems

*Bose-Einstein Condensation, Fermi Levels, Chandrasekhar Limit,...*

Normally forbidden processes

*Atomic Transitions*

# Cosmology: Post Inflation Preheating

$\mathcal{L} \propto \varphi\chi^2$ ($\varphi =$ inflaton, $\chi =$ matter)

$$\ddot{\chi} + \underbrace{\omega_0^2 \left(1 + h\cos\left[(2\omega_0 + \epsilon)t\right]\right)}_{\phi(t)} \chi = \alpha f(\chi, \dot{\chi}) \quad \text{(RHS=GUP)}$$

$$\chi(t) = a(t)\cos\left(\frac{\theta}{2}t\right) + b(t)\sin\left(\frac{\theta}{2}t\right), \quad a \sim e^{st}, b \sim e^{st}$$

*Parametric Resonance*

| | sign of $\Delta s_{GUP}$ ($\alpha > 0$) | sign of $\Delta s_{GUP}$ ($\alpha < 0$) |
|---|---|---|
| $b/a = 1$ | $+$ | $-$ |
| $b/a = 0$ | $+$ | $-$ |
| $b/a = \infty$ | $+$ | $-$ |
| $b/a \neq 1$ | $\pm$ | $\pm$ |

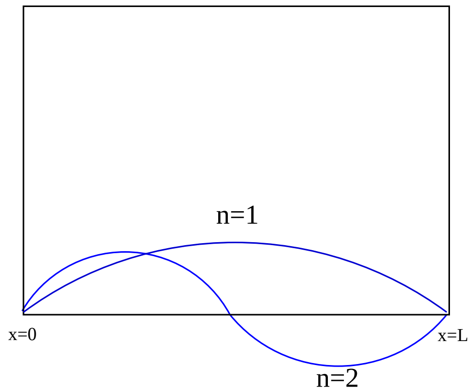
Instability region

$$\Delta\epsilon = h\omega_0 \left[1 - 2\alpha\theta\ell a^2\right] \quad \textit{(Depends on sign of } \alpha\textit{)}$$

W. Chemissany, S. Das, E. C. Vagenas, JCAP 1112:017 (2011), arXiv:1111.7288

$$\boxed{\text{Look at } \textit{New Non-perturbative} \text{ solution of the } \textit{Cubic} \text{ Schrödinger Equation}}$$

## Particle in a Box



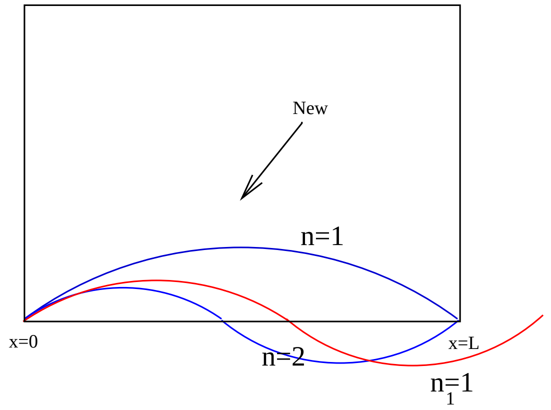

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad {\scriptstyle (k \equiv \sqrt{2mE/\hbar^2})}$$

$$\psi(0) = 0 \;\rightarrow A + B = 0 \;\rightarrow \psi(x) = 2iA\sin(kx)$$

$$\psi(L) = 0 \;\rightarrow kL = n\pi \;\rightarrow E_n = \frac{k^2\hbar^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

## Particle in a Box with GUP




$$\left[-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - i\frac{\alpha\hbar^3}{m}\frac{d^3}{dx^3}\right] = E\psi$$

$$\psi(x) = \underset{\substack{\uparrow \\ \textit{Choose Real}}}{A} e^{ik'x} + Be^{-ik''x} + \underset{\substack{\uparrow \\ \text{New Solution} \\ \textit{(Cannot just set it to zero!)}}}{C} e^{\frac{ix}{2\alpha\hbar}}$$

$$k' = k(1 + k\alpha\hbar)\ ,\ \ k'' = k(1 - k\alpha\hbar)$$

$$\psi(0) = 0 \ \rightarrow A + B + C = 0 \ \rightarrow$$

$$\psi = 2iA\sin(kx) + C\left[-e^{-ikx} + e^{ix/2\alpha\hbar}\right] - \alpha\hbar k^2 x\left[i\ Ce^{-ikx} + 2A\sin(kx)\right]$$

$$\psi(L) = 0 \ \rightarrow$$

$$\begin{aligned} 2iA\sin(kL) &= |C|\left[e^{-i(kL+\theta_C)} - e^{i(L/2\alpha\hbar-\theta_C)}\right] \underbrace{+\,\alpha\hbar k^2 L\left[i\,|C|\,e^{-i(kL+\theta_C)} + 2A\sin(kL)\right]}_{\mathcal{O}(\alpha^{\text{higher power}})} \\ &= |C|\left[e^{-i(kL+\theta_C)} - e^{i(L/2\alpha\hbar-\theta_C)}\right] \end{aligned}$$

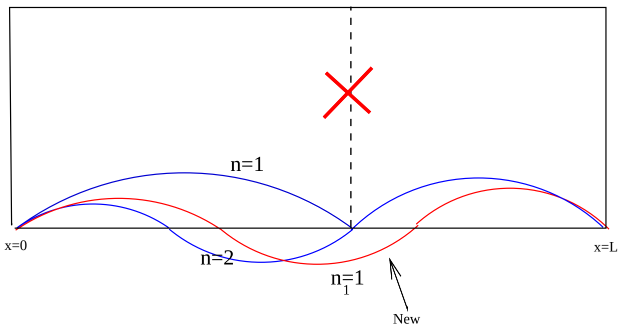
$C = |C|e^{-i\theta_C}$ etc

Take real parts of both sides ($A$ = Real)

$$\cos\left(\frac{L}{2\alpha\hbar} - \theta_C\right) = \cos(kL + \theta_C) = \cos(n\pi + \theta_C + \epsilon)$$

Solution

$$\begin{aligned} \frac{L}{2\alpha\hbar} &= \frac{L}{2\alpha_0\ell_{Pl}} = n\pi + 2q\pi + 2\theta_C \equiv n_1\pi + 2\theta_C \\ \frac{L}{2\alpha\hbar} &= \frac{L}{2\alpha_0\ell_{Pl}} = -n\pi + 2q\pi \equiv n_1\pi \\ n_1 &\equiv 2q \pm n \in N. \end{aligned}$$




Only certain $L$s can fit both $\sin(kL)$ and $\cos(\frac{L}{2\alpha\hbar})$

Box Length is Quantized!

Need at least one particle for measuring lengths

Perhaps all measured lengths are quantized?

A. Ali, S. Das, E. C. Vagenas, Phys. Lett. **B678**, 497-499 (2009), arXiv:0906.5396

## Relativistic Wave Equations

- Klein-Gordon

For Stationary States: $2mE \rightarrow E^2 - m^2$ , $k \rightarrow k\sqrt{\frac{E}{2mc^2} - \frac{mc^2}{2E}}$

**$L$ Quantization Unchanged**

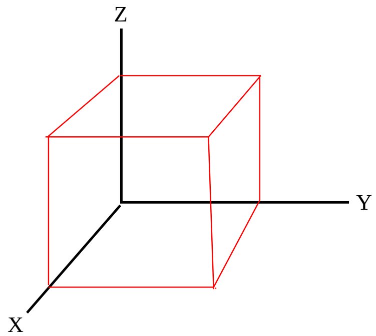
<u>Problems with KG</u>

- Most elementary particles are fermions
- How to generalize to 2 and 3 dimensional box?

$$\vec{p} = \vec{p}_0 - \alpha p_0 \vec{p} = \vec{p}_0 - \alpha\sqrt{p_{0x}^2 + p_{0y}^2 + p_{0z}^2}\ \vec{p}$$

$$\rightarrow \vec{p}_0 - \alpha\hbar \underbrace{\sqrt{-\frac{d^2}{dx^2} - \frac{d^2}{dy^2} - \frac{d^2}{dz^2}}}_{\text{Non-local}}\ \vec{p}$$

# Dirac Equation




- $p_0 \to \vec{\alpha} \cdot \vec{p}$

$$H\psi = \left(\vec{\alpha} \cdot p + \beta mc^2\right)\psi = \left(\vec{\alpha} \cdot \vec{p}_0 - \alpha(\vec{\alpha} \cdot \vec{p}_0)(\vec{\alpha} \cdot \vec{p}_0) + \beta mc^2\right)\psi = E\psi$$

$$\psi \equiv e^{i\vec{t}\cdot\vec{r}} \begin{pmatrix} \chi \\ \vec{\rho} \cdot \vec{\sigma}\chi \end{pmatrix} \to H\psi = e^{i\vec{t}\cdot\vec{r}} \begin{pmatrix} ((m - \alpha t^2) + \vec{t} \cdot \vec{r} + i\vec{\sigma} \cdot (\vec{t} \times \vec{\rho}))\chi \\ \left(\vec{t} - (m + \alpha^2 t^2)\vec{\rho}\right) \cdot \vec{\sigma}\chi \end{pmatrix} = E\psi \ ,$$

$$\psi = e^{i\vec{k}\cdot\vec{r}} \begin{pmatrix} \chi \\ r\hat{k} \cdot \vec{\sigma}\chi \end{pmatrix} \ , \quad \psi = e^{i\frac{\hat{q}\cdot\vec{r}}{\alpha\hbar}} \begin{pmatrix} \chi \\ \hat{q} \cdot \vec{\sigma}\chi \end{pmatrix}$$

## Confining Wavefunction

Superposition of $2^d + 1$ eigenfunctions

(d=1,2,3)

$$\psi = \begin{pmatrix} \left[\prod_{i=1}^{d} \left(e^{ik_i x_i} + e^{-i(k_i x_i - \delta_i)}\right) + F e^{i\frac{\hat{q}\cdot\vec{r}}{\alpha\hbar}}\right]\chi \\ \sum_{j=1}^{d} \left[\prod_{i=1}^{d} \left(e^{ik_i x_i} + (-1)^{\delta_{ij}} e^{-i(k_i x_i - \delta_i)}\right) r\hat{k}_j \right. \\ \left. + F e^{i\frac{\hat{q}\cdot\vec{r}}{\alpha\hbar}} \hat{q_j}\right] \sigma_j \chi \end{pmatrix}$$

## MIT Bag Boundary Conditions

(Zero flux through boundaries)

$$\bar{\psi}\gamma^{\mu}\psi = 0 \;\leftrightarrow\; \pm i\beta\alpha_l\psi = \psi$$

$$e^{i\delta_k}\left(1 + ir\hat{k}_k\right) = \left(ir\hat{k}_k - 1\right) + f_{\bar{k}}^{-1} F'_k e^{-i\theta_k} \quad (x_k = 0)$$

$$e^{i(2k_k L_k - \delta_k)}\left(1 + ir\hat{k}_k\right) = \left(ir\hat{k}_k - 1\right) + f_{\bar{k}}^{-1} F'_k e^{i\left(\frac{\hat{q}_k L_k}{\alpha\hbar} + \theta_k\right)} e^{i(k_k L_k - \delta_k)} \quad (x_k = L_k)$$

$$\left[F'_k \equiv \sqrt{1 + |\hat{q}_k|^2} F \; , \; \theta_k \equiv \arctan \hat{q}_k\right]$$

Comparing

$$k_k L_k = \delta_k = \arctan\left(-\frac{\hbar k_k}{mc}\right) + \mathcal{O}(\alpha)$$

$$\boxed{\frac{L_k}{\alpha_0 \ell_{Pl}} = (2p_k\pi - 2\theta_k)\sqrt{d} \; , \quad p_k \in N}$$

$$\left[|\hat{q}_k| = 1/\sqrt{d}\right]$$

$$A_N \equiv \prod_{k=1}^{N} \frac{L_k}{\alpha_0 \ell_{Pl}} = d^{N/2} \prod_{k=1}^{N} (2p_k\pi - 2\theta_k) \; , \quad p_k \in N \; .$$

**Measurable Lengths, Areas and Volumes are Quantized!**


A. Ali, S. Das, E. C. Vagenas, Phys. Lett. **B690** (2010) 407-412, arXiv:1005.3368

P. Alberto, S. Das, E. C. Vagenas, Phys. Lett. **A375** (2011) 1436-1440, arXiv:1102.3192

## Generalizations in curved spacetimes

- Work in progress with S. Deb

## Self-adjointness of GUP modified operators

Operator $\hat{A}$

$\mathcal{K}_{+} = \ker(i - \hat{A}^{\dagger})$
$\mathcal{K}_{-} = \ker(i + \hat{A}^{\dagger})$

$n_{+} = \dim \left[\mathcal{K}_{+}\right] \, \& n_{-} = \dim \left[\mathcal{K}_{-}\right]$

(Deficiency indices)

**von-Neumann's Theorem**

Let $\hat{A}$ be a closed symmetric operator with deficiency indices $n_{+}$ and $n_{-}$. Then,

- $\hat{A}$ is self-adjoint if and only if $(n_{+}, n_{-}) = (0, 0)$
- $\hat{A}$ has self-adjoint extensions if and only if $n_{+} = n_{-}$. These extensions are parametrized by an $n$ x $n$ unitary matrix
- If $n_{+} \neq n_{-}$, the $\hat{A}$ has no self-adjoint extensions

V. Bala et al (2011-)

## Appendix 3A

### 3.3.5 Momentum Operator: Quadratic in $\alpha$

$$\begin{aligned}\hat{p} &= \hat{p}_0 + 2\alpha^2\hat{p}^3 \\ &= i2\alpha^2\hbar^3\frac{d^3}{dx^3} - i\hbar\frac{d}{dx}\end{aligned} \tag{3.3.12}$$

### 3.3.6 Deficiency Indices: $n_+ = \dim\left[\ker\left(\frac{i}{\alpha} - \hat{p}^\dagger\right)\right], \psi_+(x)$

**Equation**

$$\begin{aligned}\hat{p}^\dagger\psi_+(x) &= \frac{i}{\alpha}\psi_+(x) \\ i2\alpha^2\hbar^3\frac{d^3\psi_+(x)}{dx^3} - i\hbar\frac{d\psi_+(x)}{dx} - \frac{i}{\alpha}\psi_+(x) &= 0 \\ 2\alpha^3\hbar^2\frac{d^3\psi_+(x)}{dx^3} - \frac{d\psi_+(x)}{dx} - \frac{1}{\alpha}\psi_+(x) &= 0\end{aligned} \tag{3.3.13}$$

**Characteristic Equation**

$$2\alpha^2\hbar^2m^3 - m - \frac{1}{\alpha} = 0 \tag{3.3.14}$$

**Roots**

$$m_1 = \frac{1}{\alpha\hbar} \tag{3.3.15}$$

$$m_2 = \frac{1}{\alpha\hbar}\left(\frac{-1}{2} + \frac{i}{2}\right) \tag{3.3.16}$$

$$m_3 = \frac{1}{\alpha\hbar}\left(\frac{-1}{2} - \frac{i}{2}\right) \tag{3.3.17}$$

Linearly independent solutions:

$$\psi_+^1(x) \propto \exp\left(\frac{x}{\alpha\hbar}\right) \tag{3.3.18}$$

$$\psi_+^2(x) \propto \exp\left[\left(\frac{-1}{2} + \frac{i}{2}\right)\frac{x}{\alpha\hbar}\right] \tag{3.3.19}$$

$$\psi_+^3(x) \propto \exp\left[\left(\frac{-1}{2} - \frac{i}{2}\right)\frac{x}{\alpha\hbar}\right] \tag{3.3.20}$$

Deficiency Index, $n_+$

Table 2.1: Results: GUP modified Momentum operator

| **Operator** | $\psi_+(x)$ | $\psi_-(x)$ | $(n_+, n_-)$ | | |
|---|---|---|---|---|---|
| | | | $(-\infty, \infty)$ | $[0, \infty)$ | $[a, b]$ |
| $\hat{p}_0$ | $A_1 \exp\left[-\frac{x}{d}\right]$ | $A_2 \exp\left[\frac{x}{d}\right]$ | $(0, 0)$ | $(1, 0)$ | $(1, 1)$ |
| $\hat{p}_0(1 - \alpha \hat{p}_0)$ | $B_1 \exp[\gamma_1 x]$ | $B_3 \exp[\gamma_3 x]$ | $(0, 0)$ | $(1, 1)$ | $(2, 2)$ |
| | $B_2 \exp[\gamma_2 x]$ | $B_4 \exp[\gamma_4 x]$ | | | |
| $\hat{p}_0 \left(1 + 2\alpha^2 \hat{p}_0^2\right)$ | $C_1 \exp[\beta_1 x]$ | $C_4 \exp[\beta_4 x]$ | $(0, 0)$ | $(2, 1)$ | $(3, 3)$ |
| | $C_2 \exp[\beta_2 x]$ | $C_5 \exp[\beta_5 x]$ | | | |
| | $C_3 \exp[\beta_3 x]$ | $C_6 \exp[\beta_6 x]$ | | | |
| $\hat{p}_0 \left(1 - \alpha \hat{p}_0^2 + 2\alpha^2 \hat{p}_0^3\right)$ | $D_1 \exp[\lambda_1 x]$ | $D_4 \exp[\lambda_4 x]$ | $(0, 0)$ | $(2, 1)$ | $(3, 3)$ |
| | $D_2 \exp[\lambda_2 x]$ | $D_5 \exp[\lambda_5 x]$ | | | |
| | $D_3 \exp[\lambda_3 x]$ | $D_6 \exp[\lambda_6 x]$ | | | |

Table 2.2: Results: GUP modified Hamiltonian operator

| **Operator** | $\psi_+(x)$ | $\psi_-(x)$ | $(n_+, n_-)$ | | |
|---|---|---|---|---|---|
| | | | $(-\infty, \infty)$ | $[0, \infty)$ | $[a, b]$ |
| $\frac{1}{2m}\hat{p}_0^2$ | $a_+ \exp[k_+ x]$ | $a_- \exp[k_- x]$ | $(0, 0)$ | $(1, 1)$ | $(2, 2)$ |
| | $b_+ \exp[-k_+ x]$ | $b_- \exp[-k_- x]$ | | | |
| $\frac{1}{2m}\hat{p}_0^2 - \frac{\alpha}{m}\hat{p}_0^3$ | $\tilde{B}_1 \exp[m_1 x]$ | $\tilde{B}_4 \exp[m_4 x]$ | $(0, 0)$ | $(1, 2)$ | $(3, 3)$ |
| | $\tilde{B}_2 \exp[m_2 x]$ | $\tilde{B}_5 \exp[m_5 x]$ | | | |
| | $\tilde{B}_3 \exp[m_3 x]$ | $\tilde{B}_6 \exp[m_6 x]$ | | | |
| $\frac{1}{2m}\hat{p}_0^2 + \frac{5\alpha^2}{2m}\hat{p}_0^4$ | $\tilde{C}_1 \exp[\kappa_1 x]$ | $\tilde{C}_5 \exp[\kappa_5 x]$ | $(0, 0)$ | $(2, 2)$ | $(4, 4)$ |
| | $\tilde{C}_2 \exp[\kappa_2 x]$ | $\tilde{C}_6 \exp[\kappa_6 x]$ | | | |
| | $\tilde{C}_3 \exp[\kappa_3 x]$ | $\tilde{C}_7 \exp[\kappa_7 x]$ | | | |
| | $\tilde{C}_4 \exp[\kappa_4 x]$ | $\tilde{C}_8 \exp[\kappa_8 x]$ | | | |
| $\frac{\hat{p}_0^2}{2m} - \frac{\alpha}{m}\hat{p}_0^3 + \frac{5\alpha^2}{2m}\hat{p}_0^4$ | $\tilde{D}_1 \exp[\eta_1 x]$ | $\tilde{D}_5 \exp[\eta_5 x]$ | $(0, 0)$ | $(2, 2)$ | $(4, 4)$ |
| | $\tilde{D}_2 \exp[\eta_2 x]$ | $\tilde{D}_6 \exp[\eta_6 x]$ | | | |
| | $\tilde{D}_3 \exp[\eta_3 x]$ | $\tilde{D}_8 \exp[\eta_7 x]$ | | | |
| | $\tilde{D}_4 \exp[\eta_4 x]$ | $\tilde{D}_8 \exp[\eta_8 x]$ | | | |

## Proposed Experiments

- Quantum Optomechanics, Gravity Bar Detectors
- Superplanckian *quantum* centre-of-mass mode:
  $[x, p] = i\hbar\,(1 + \cdots)$ $\leftarrow$ GUP
- Cryogenic

## Proposed Experiment 1: Quantum Optics

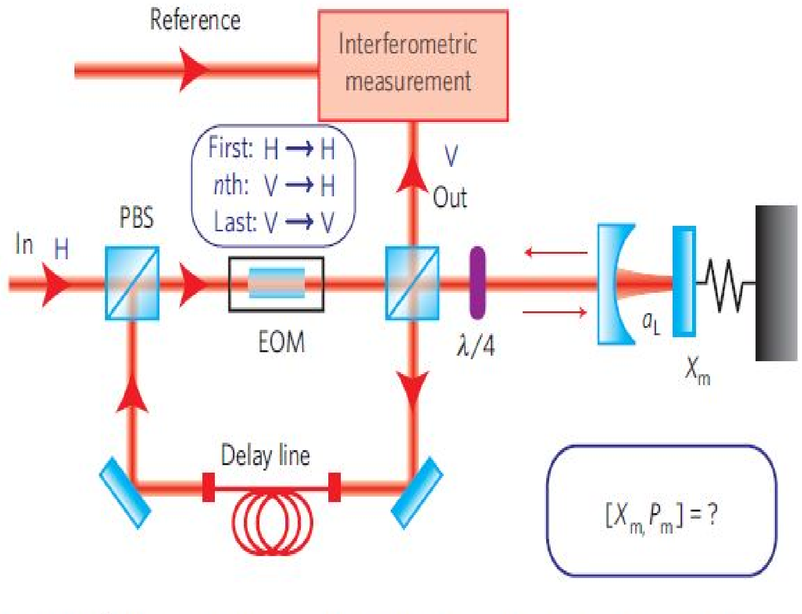



**Figure 3** | Proposed experimental set-up to probe deformations of the canonical commutator of a macroscopic mechanical resonator. An

Experiment and figure: I. Pikovski et al, **Nature Physics**, Vol.8 (2012), 393-397 (*U Vienna, Blackett Laboratory*)

Displacement operator

$$\xi = e^{i\lambda n_L P_m} e^{-i\lambda n_L X_m} e^{-i\lambda n_L P_m} e^{i\lambda n_L X_m}$$

$$\exp\left(-i\lambda n_L \sum_k (\lambda n_L)^k C_k/k\right)$$

$$[X_m, P_n] = iC_1 \ , \quad iC_k = [X_m, C_{k-1}] \ \leftarrow GUP$$

Mean optical field $\langle a_L \rangle \propto \exp\left(-\frac{4}{3}\beta N_p^3 \lambda^4 e^{-i6\lambda^2}\right)$

# Proposed Experiment 2: Gravity Bar Detectors

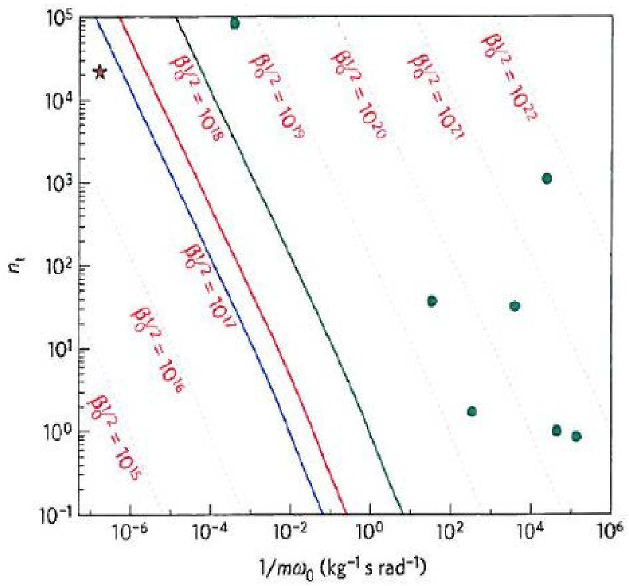



**Figure 1 | Upper limits to the parameter $\beta_0$ that quantifies the deformation of the standard uncertainty relation in equation (2).** The star corresponds to the AURIGA experiment discussed in the main text, and

Experiment and figure: F. Marin et al, **Nature Physics**, Vol.9 (2013), 71-73 (Italy, Netherlands)

$$E_{min} = \frac{\hbar\omega_0}{2}\left[\sqrt{1+\frac{\beta^2}{4}}+\frac{\beta}{2}\right]$$

## Summary and Conclusions

- One GUP seems to fit Black Holes, String Theory, DSR,...
- GUP affects all QM Hamiltonians. At least 1 part in $10^{12}$ precision required for measuring effects
- Space Quantized near the Planck scale. *But*, Discreteness at $10^{-35}\ m \ \rightarrow$ observable effects at $10^{-20}\ m$?
- Applications to cosmology: B. Majumdar (many papers), Palma & Patil, Zhu, Ren & Li
- Keplerian orbits, Thermodynamics, Holography, Equivalence Principle, Electroweak Theory, Black Holes, Information Problem, Higher dimensions, LHC
- Further applications: theory and experiments
- Proposed experiments to measure modified $[x, p]$ commutator
- Optimistic Scenario: A Low Energy Window to Quantum Gravity Phenomenology?